Exploring Rainbows

0.0.1 Lab Ticket

Read this lab write-up and write an outline of the procedure you will follow during your two-week investigation.

Your lab write-up should summarize all of the investigations you undertook in the lab. Include all important plots and findings; motivate your arguments and discuss the method you used to come to your results. Be sure and discuss possible sources of random or systematic error. Feel free to explore questions not specifically asked in the lab (in fact, it is encouraged).

0.1 Introduction

Virtually every culture has legends about or including the rainbow. For millennia, scientists and laymen alike have admired, studied and honored in verse and song the phenomenon of the rainbow. Yet the basic physical explanation of rainbows was not discovered until Descartes, equipped with the theory of ray optics, analyzed sunlight incident on a drop of water in 1637. Descartes' theory correctly predicted the location of a rainbow, but could not explain the range of colors seen. The colors of a rainbow were not understood until Newton explained that white light was made up of all colors, and that the colors separate when passing through a drop because of their differing indices of refraction.

In this lab you will analyze the rainbow from a ray optics perspective. From this treatment you will be able to explore the fundamental features of rainbow phenomenon. In particular, you will determine and measure the multiple "rainbow angles" discovered by Descartes and verify the dispersion of light through water.

0.2 Theory

The ray optics theory of rainbows relies heavily on three properties of the interaction of light and matter; refraction, reflection and dispersion. Reflection and refraction are used to explain the way in which a ray of light travels through a raindrop, and are responsible for the location of a rainbow. Dispersion explains how white light is separated into a rainbow of colors.

0.2.1 Reflection and Refraction

Consider a spherical drop of water illuminated with sunlight. Some of the light will be reflected off the drop's outer surface (Fig. 1a) and the rest will refract into the drop, cross the drop and mostly refract out (Fig. 1b). The white light following the later path does disperse to some extent but the

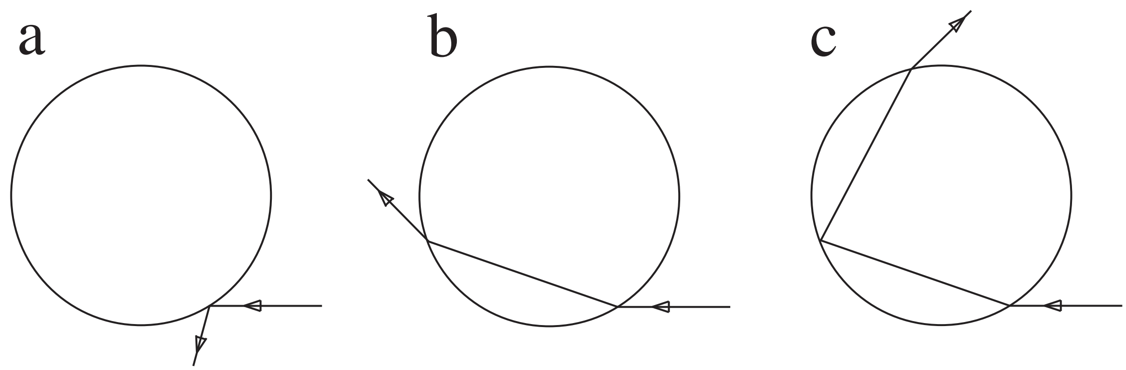

Figure 1: **a)** Light reflected on the outer surface, **b)** travelling through the drop once and then exiting or **c)** travelling through the drop and being internally reflected once (as pictured here) or more before exiting.

geometry of the drop makes it so that the exiting light is essentially white. A fraction of the light within the drop will internally reflect off the boundary of the drop, travel further through the drop and then refract out (Fig. 1c). The light exiting at this point is the source of the brightest, most often seen rainbow in the sky, the "primary rainbow."

The light can also undergo another internal reflection before exiting the drop. This light is responsible for the "secondary rainbow," which can also be seen in nature. Of course the process doesn't stop here. In theory smaller and smaller fractions of the light undergo any number of internal reflections and, with the right equipment, upwards of 15 rainbows can be seen. In nature only the first two are visible, due to the intensity of background light. In this lab, you should easily be able to view four rainbows.

Descartes' contribution to the theory of rainbows was in determining the "rainbow angle," the angle at which the various rainbows exit the drop relative to the incident sunlight angle. The incident sunlight angle, θ_i , is the angle the incoming sunlight makes with the boundary of the drop (Fig. 2b). Descartes correctly reasoned that the rainbow angle would be the angle at which the exiting light is most intense. The intensity of emerging light is greatest around the ray which has been deviated *least* by the drop, relative to the direction of the incident sunlight (Fig. 2a).

We start calculating the rainbow angle by finding θ, the angle of light exiting the drop relative to θ_i, as a function of θ_i (Fig. 2b). Upon entering a drop at an incident angle θ_i the light ray is deviated by an angle $\theta_i - \theta_r$ where θ_i and θ_r are related by Snell's law:

$$n_{air} \sin[\theta_i] = n_{water} \sin[\theta_r]. \tag{1}$$

An internal reflection then causes the ray to be deviated by an angle $180 - 2\theta_r$ and is finally turned by an angle $\theta_i - \theta_r$ when leaving the drop. Thus the total deviation of a ray internally reflected once is

$$\theta = 180^\circ + 2\theta_i - 4\theta_r. \tag{2}$$

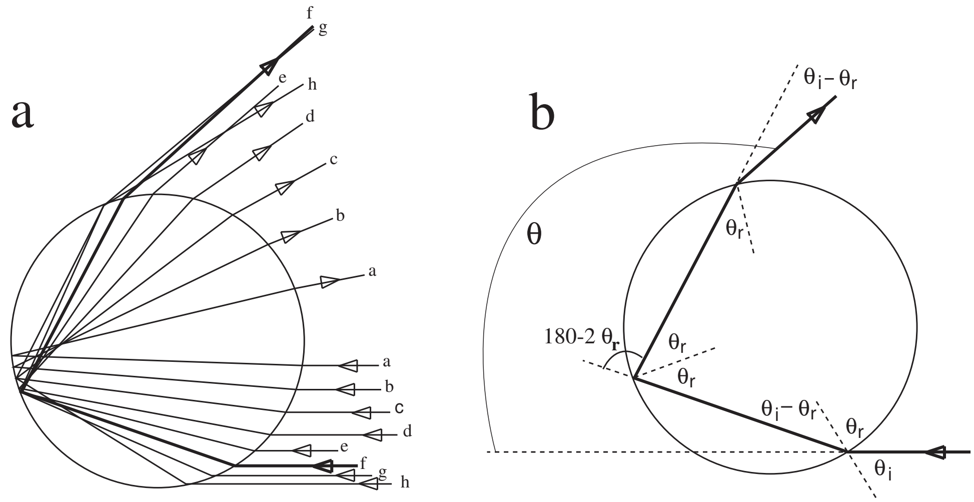

Figure 2: **a)** Rays experiencing a single internal reflection. The angle at which the light ray which deviates the least (smallest θ) is the "rainbow angle" for that order of rainbow. Intensity is greatest around that ray (ray "f" in this case). **b)** Geometry of a ray undergoing one internal reflection before exiting the drop. The deviation the ray experiences, θ, is equal to the sum of each angular deviation given by Eq. (2).

Using Eq. (1), θ_r can be written in terms of θ_i:

$$\theta_r = \arcsin\left[\frac{\sin[\theta_i]}{n_{water}}\right], \tag{3}$$

with the very good approximation that $n_{air} = 1$. Plugging Eq. (3) into Eq. (2) gives

$$\theta = 180^\circ + 2\theta_i - 4\arcsin\left[\frac{\sin[\theta_i]}{n_{water}}\right]. \tag{4}$$

To validate Descartes' reasoning that light is the most intense around the ray that deviates the least, plot θ as a function of θ_i and convince yourself that there is a value of θ_i for which a given change in θ_i results in a minimum change in θ (use n=1.331). Assuming that each incident angle receives the same intensity of light, you have therefore convinced yourself that there is such a θ_i at which light emerges more intensely than at other θ_is. As you can see from the plot, this point just happens to be when θ attains a minimum. Include this plot and analysis in your lab write-up, along with an argument explaining why the ray deviated the least should be in the area of most intense light.

To find the minimum of Eq. (4), since we want to find the smallest value of θ, take its derivative with respect to θ_i and set the result equal to zero:

$$\frac{\partial\theta}{\partial\theta_i} = 2 - \frac{4\cos[\theta_i]}{n_{water}\sqrt{1 - \frac{\sin[\theta_i]^2}{n_{water}^2}}} = 0. \tag{5}$$

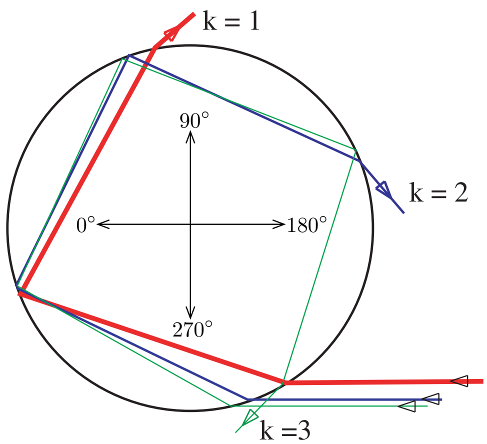

Figure 3: An diagram of the approximate exiting points of the first three order rainbows. The red, thick line represents the primary rainbow, the blue line is the secondary and the thin green line is the tertiary. Light enters the drop from the right, and the compass in the center indicates the rainbow angle, which is the deviation from the original path.

After simplification, θ_i is determined by the relationship

$$\cos[\theta_i] = \left(\frac{1}{3}(n_{water}^2 - 1)\right)^{1/2}. \tag{6}$$

Finally, to find the rainbow angle for the primary rainbow, θ_{prim}, we solve for θ_i and plug the result[1] into Eq. (2), yielding

$$\theta_{prim} = 180^\circ + 2\arccos[\left(\frac{1}{3}(n_{water}^2 - 1)\right)^{1/2}] - 4\arcsin[\frac{\sin[\arccos[\left(\frac{1}{3}(n_{water}^2 - 1)\right)^{1/2}]]}{n_{water}}]. \tag{7}$$

Using the index of refraction for red light[2], $n_{water} = 1.331$, we obtain $\theta_i \approx 59.5^\circ$ and θ_{prim} is

$$\theta_{prim} \approx 137.6^\circ. \tag{8}$$

The process for determining the rainbow angle for the secondary (θ_{sec}), tertiary (θ_{tert}), and in general, the k^{th} order rainbow angle is similar. For k internal reflections, Eq. (2) becomes

$$\theta = k180^\circ + 2\theta_i - 2(k+1)\theta_r \tag{9}$$

and equation (6) is then

$$\cos[\theta_i] = \left(\frac{n_{water}^2 - 1}{k(k+2)}\right)^{1/2}. \tag{10}$$

[1]Symbolically, θ_{prim} could be described as the point at which $\frac{\partial\theta}{\partial\theta_i}|_{\theta_i=\theta_{prim}} = 0$.

[2]$\lambda \approx 700nm$. Red light is used for practical reasons to be explained later.

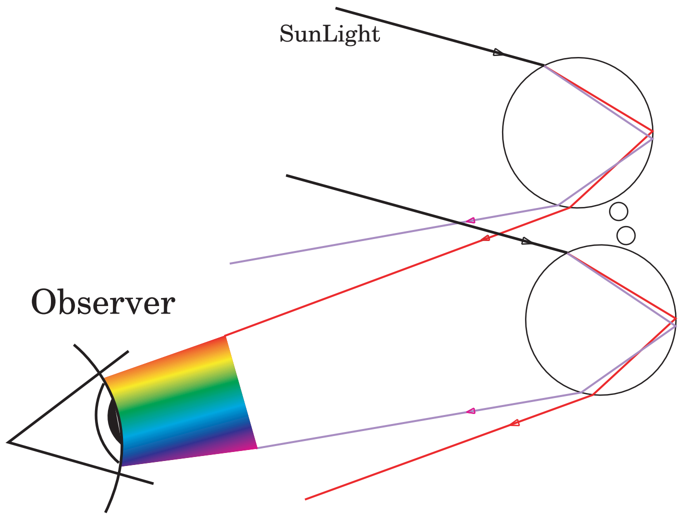

Figure 4: Natural rainbows are formed by light refracting and dispersing in many droplets. The droplets at the "top" of the rainbow are in such a relation to the sun that they contribute red light while the droplets at the "bottom" of the rainbow contribute blue/voilet light. Droplets in-between the top and bottom are responsible for the colors in-between red and blue/violet.

Using the red wavelength value of the index of refraction of water, the results for the secondary and tertiary rainbow angles are

$$\begin{aligned} \theta_i &= 71.9^\circ \quad \Rightarrow \quad \theta_{sec} = 230.4^\circ \\ \theta_i &= 76.9^\circ \quad \Rightarrow \quad \theta_{tert} = 317.5^\circ, \end{aligned} \tag{11}$$

respectively. It is important to note that not only does the rainbow angle θ become larger with each successive order of rainbow, but so does the incident angle θ_i, with the upper limit being 90°. To give an idea of how much θ_i can vary, the 20^{th} rainbow has a θ_i of $\approx 87.6^\circ$.

0.2.2 Dispersion

While Descartes, using reflection and refraction, correctly explained the manner in which light enters, travels through and exits a raindrop at certain angles, his explanation of why there are colors at these angles was wrong.[3] Newton postulated the correct reason, theorizing that sunlight was actually made of a combination of light of various colors, and that the index of refraction of water n_{water} is slightly different for different colors. This would mean that each color would have a slightly different rainbow angle and hence the exiting light would be spread into a spectrum of colors. For example, to calculated the angular spread of the primary rainbow, subtract θ_{blue} (Eq. (7), using the blue light index of refraction in water) from θ_{red} (Eq. (7), using the red light index of refraction in water). This difference in rainbow angles, $\Delta\theta$, comes out to be around 1.72°, the angular width of the primary rainbow.

[3]Descartes thought that the "qualitative change" (i.e. color) light underwent when travelling through matter was a function of how fast the "subtle" matter (what we think of as molecules) was rotating.

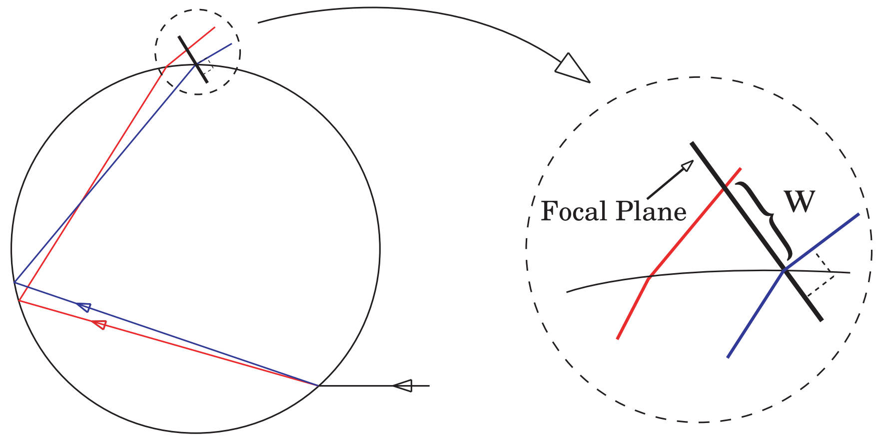

Figure 5: A picture exaggerating the path differences of red and blue rays in the bulb. The width of the rainbow, w, is defined to be the path difference between the red and blue light perpendicular to the path of the blue light at the point where the blue light exits the bulb. This perpendicular line, or rather in three dimensions, plane, is the desired plane on which to focus the telescope piece.

Due to the nature of the setup you will be using, expressing the dispersion of white light through the raindrop in terms of $\Delta\theta$ is not useful. Instead, express the dispersion as the physical distance, w, the light spreads when exiting the drop (Fig. 5).

The simplest way to determine w analytically is to mathematically trace out the red and blue light rays and calculate the difference in their paths in a plane normal to the line of sight (Fig. 5). But again, there are experimental complications. To measure the dispersion of the rainbow, you will be using a telescope eyepiece that has a particular focal length. The red and blue light rays exit the drop far enough apart that either the red or the blue light ray can be well focused, but not both. Focusing on either ray would be acceptable if not for the fact that, in general, one color will dominate others if the drop is observed at that color's rainbow angle. At the rainbow angle for blue light, the emerging light has a mixture of all visible colors, but because the blue rays are most intense at that angle, blue dominates. Viewing rays at red light's rainbow angle is a bit different. Since red light has the smallest index of refraction its rainbow angle is the smallest. The geometry of the drop works out so that no blue light exits the water drop at that angle, hence only red is seen at red's rainbow angle. Therefore to see the entire rainbow, we must look at and focus on blue's rainbow angle.

The calculation is straightforward and only requires Snell's law and some simple geometry, but is somewhat tedious so the details will be omitted. The result is that the width of the primary rainbow w is essentially linear with respect to the radius of the drop R:

$$w = 0.01586\, R, \tag{12}$$

where R is measured in meters. The spread of rainbows beyond the primary rainbow is difficult to see with our set-up and will be left to those who are interested.

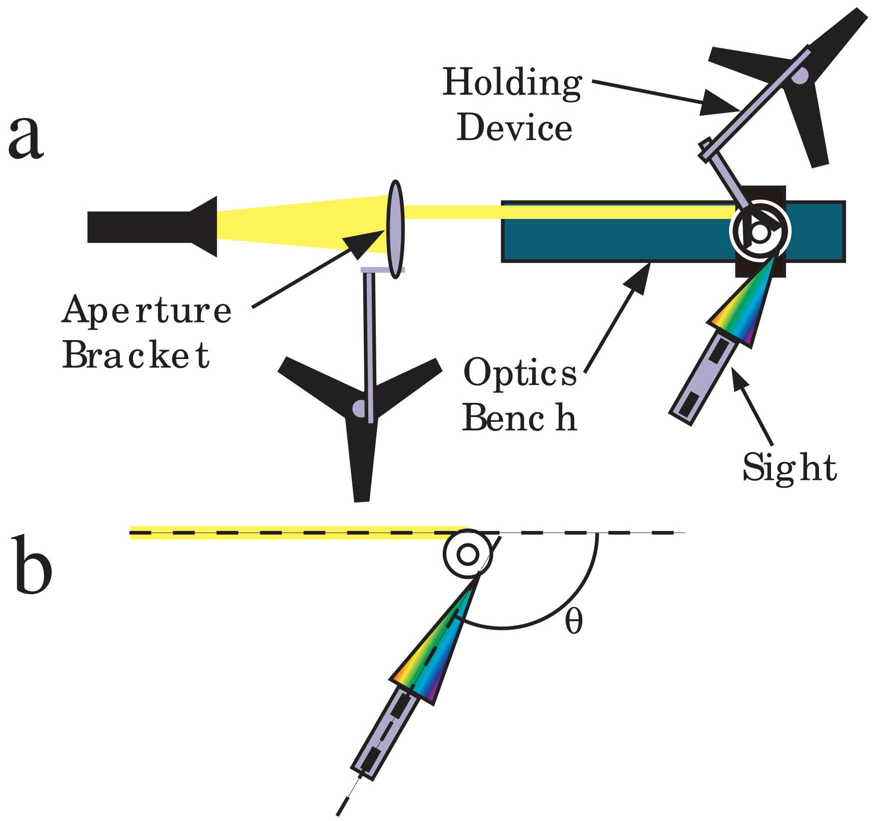

Figure 6: **a)** A detailed diagram of the entire experimental setup, including the light source (far left), aperture bracket, optics table (attached to the optics bench), glass sphere, and sight. **b)** A simplified diagram of the setup showing only the incoming light, glass sphere, and outgoing rainbow. The rainbow angle, θ, measures the deviation of the rainbow from the original path of the incoming light.

0.3 Experimental Procedure

The setup we will use to explore rainbows in the lab is almost identical to that used by Descartes, over 350 years ago. This setup consists primarily of a small glass sphere filled with water, standing in for a raindrop, and a flashlight, standing in for the sun. A thin beam of light will enter the drop, refract/reflect along its path through the drop, and then exit as a small colorful spectrum. Once the angular position of the rainbow spectrum has been determined, you'll take a picture of the setup from overhead, and use *VideoPoint* to determine the rainbow angle for several rainbows. The setup is depicted in Fig. 6.

The glass sphere will be held directly over a PASCO optics table, which is mounted on the optics bench. On the underside of the optics table is mounted a sighting device that will allow you to accurately measure the rainbow angle. The flashlight should be mounted at the same level as the bulb, and be aimed at one side of the bulb. An aperture bracket set to the large slit allows only a narrow, nearly parallel beam of light to reach the edge of the bulb. This beam should be as bright as possible to make several rainbows visible, but narrow enough to prevent excessive glare.

0.3.1 Rainbow Angles

We are now ready to determine the rainbow angle. First you must locate the primary rainbow. It will be located on the edge of the bulb, roughly opposite the point where the light enters the bulb. Position your eye level with the middle of the bulb and scan around the edge of the bulb until you see the primary rainbow. You may see several small bands of color or glare; the primary rainbow will be considerably brighter and its colors will be more spread out. Once you have found the primary rainbow, use the sight to determine its position more accurately. With the rainbow in view, move the outer arm of the sight so that the red portion of the rainbow is visible through the notch in the sight. Remember that our theoretical rainbow angles are for red ($\lambda = 700$nm) light, so you should be focusing on the very outer edge of the red band. Next, position the inner arm of the sight so that it blocks the red portion of the spectrum that was visible through the notch.

Now that you've marked the position of the primary rainbow with the sight, record the setup with a camcorder and analyze the clip in *VideoPoint*. Using a tripod, position the camera about .75m above the setup, aimed straight down. To minimize possible perspective problems, center the field of view on the inner arm of the sight. Make sure that there is enough information about the setup in the screen so that you will be able to determine the rainbow angle accurately. In general you will need the tip of the light and aperture bracket to determine the direction of the incoming light, both ends of the sight to determine the angle of the outgoing light, and the bulb as a reference point. Once everything is aligned appropriately, take a very short video of the setup and import it into *VideoPoint* (you only need one frame).

Your next task is to actually determine the rainbow angle. This can be tricky, so be patient. It will help to sketch the setup as it appears on the monitor so that you can record the relevant angles as you measure them, since you probably won't be able to determine the rainbow angle directly. There are four objects to be selected in your single frame movie: the tip of the light, the slit in the aperture bracket, and the two ends of the sight (Fig. 7). With these four points marked, translate/rotate the axes so that they are aligned with the path of the incoming light beam. Record the angle of your axes, listed in the "Coordinate System" window, on your diagram of the setup. Now translate/rotate the axes so that they mark the path of the outgoing red light beam. Record this angle on your diagram. Now use geometry to determine the rainbow angle.

Find the rainbow angle for as many rainbows as you can see. With this setup it is typically possible to see the the first five rainbows. The tertiary and quaternary rainbows may be difficult to spot because they very nearly line up with spots of glare (bright white light). The quinary rainbow is faint, but visible in a dark room.

0.3.2 Rainbow Dispersion

Now you will view the primary rainbow directly and measure its dispersion as it leaves the bulb. Getting the telescope eyepiece at the right angle and distance away from the bulb can be tricky, so ask a TA for help if necessary.

Use a ruler to determine the scale of the demarcations in the eyepiece. Adjust the eyepiece so that it is level with the spherical center of the bulb. With your eye, find the approximate primary rainbow angle and keep your eye focused on the rainbow. Once you've found it, slowly move the eyepiece in front of your eye and point it at the edge of the bulb where you saw the rainbow.

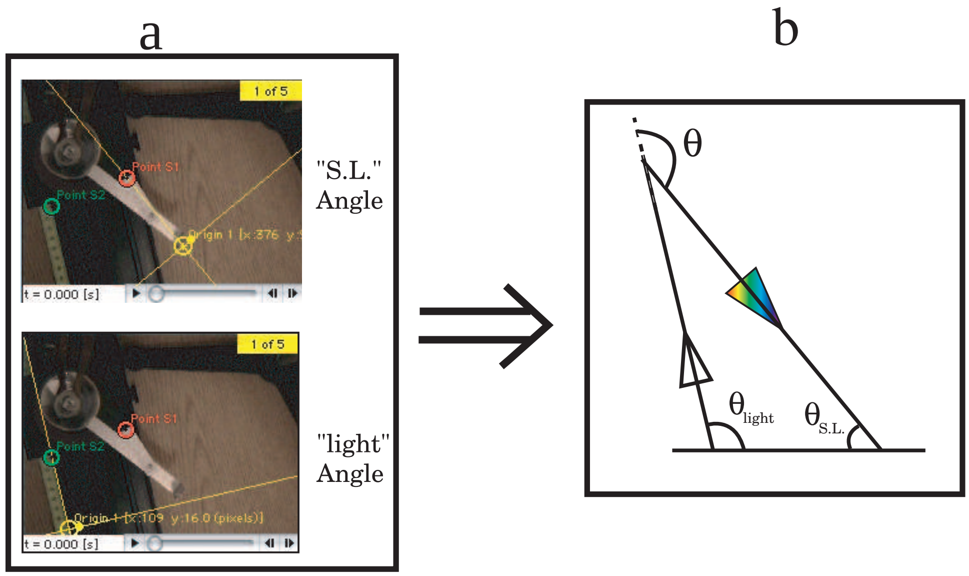

Figure 7: **a)** Screen shots of the video in *VideoPoint* which determine the angles of the incoming beam, $\theta_{S.L.}$, and outgoing rainbow, θ_{light}. **b)** The corresponding triangle.

At first it will be difficult to tell what is the rainbow and what is stray light and glare. The rainbow should be a relatively thin, relatively tall and slightly arched (following the curve of the bulb) strip of light. You may see a blob of light with a rainbow spectrum. This could very well be the rainbow, only you are not viewing it in focus. Move the eyepiece toward or away from the bulb to bring the rainbow in focus. When you get the rainbow strip in focus, move the eyepiece around the bulb (in an angular fashion) until you can see all colors of the rainbow. If the angle is slightly to large, the blue/violet portion of the rainbow will be spread out. You are looking for thin, bright, and well focused rainbow that shows all colors. Once you have found this rainbow, measure its width, from the far end of the red to the far end of the blue, using the demarcations.

0.4 Analysis

Use Eqs. (3), (9), and (10) to determine the theoretical rainbow angles for each order rainbow that you observed. Compare these theoretical rainbow angles to your measurements. What are the most significant sources of error?

Once you have measured the width of the rainbow, compare it to the theoretical width, given by Eq. (12). Calculate the primary rainbow angle for blue light ($n_{water} = 1.343$ at $\lambda = 450$nm) and for red light ($n_{water} = 1.331$ at $\lambda = 700$nm). What is the angular width of the primary rainbow? Is this reasonable, given your measurement of w? How do the widths of the higher order rainbows compare to the width of the primary rainbow?

0.5 Ideas for Independent Studies

There are many directions in which the study of rainbows can be taken. Some of the most accessible include:

- Test equation (12) by using bulbs of varying radii.
- Use your footage, geometry and Snell's law to trace the rainbow rays back to when they first entered the bulb and calculate θ_i.
- Use different transparent liquids with different indexes of refraction, such as rubbing alcohol (average index of refraction $n = 1.377$), and measure their rainbow angles. Also, use the generalized equations and techniques to calculate the angles theoretically.
- Study the polarization of the rainbows (see reference 1).
- Use the light sensor to determine the intensity of the third rainbow and see if it is indeed below the normal background light intensity.
- Study rainbows in other mediums such as CDs, glass, etc.
- Observe the order of the colors in various rainbows. How can this ordering be explained.
- Create and observe a "real" rainbow (outside). Is it consistent with your other experimental observations?

0.6 References

Here are three excellent sources to learn more about the physics of the rainbow.

1. Walker, J., "Multiple rainbows from single drops of water and other liquids," *American Journal of Physics*, **44**, 421-433, 1976.
2. Boyer, C., *The Rainbow: From Myth to Mathematics*, Thomas Yoseloff, London, 1959.
3. Greenler, R., *Rainbows, halos, and glories*, Cambridge University Press, New York, 1980.