

# Vibrational Modes of Interacting Masses

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## Lab Ticket

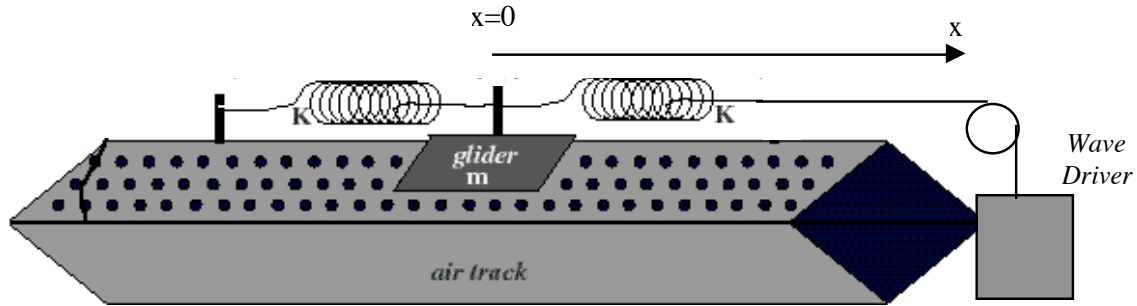
- Read this lab write-up
  - Write an outline of the procedure you will follow during your two-week investigation.
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Our physical world is rich in phenomena that can be described by the mathematics of oscillatory motion. For example, the spring-like character of chemical bonds allows the atoms in a molecule to vibrate in certain “natural” patterns, each with its own “natural frequency.” If a molecule consists of  $N$  number of atoms, it will have  $N$  possible natural modes of oscillation and thus  $N$  natural frequencies. If that molecule then is illuminated by light whose frequency coincides with one of these natural frequencies, the light will drive the molecule into the associated pattern of vibration and the light will be absorbed by the molecule. Thus by determining the natural frequencies of a molecule, one can make predictions about its optical properties (e.g., which frequencies of light it will absorb).

In this experiment, we will investigate the oscillatory patterns of an easy-to-observe system: masses moving on a nearly frictionless surface under the influence of spring forces (Hooke’s Law). To minimize frictional effects, our masses (called *gliders*) will travel on an air track, supported by tiny jets of air, just like a puck in an amusement-hall air hockey game. In turn, we will explore the vibrational modes for one, two, and finally three, masses connected together by springs.

## Theory of Oscillatory Motion in Presence of Driving Force

As a first step, consider one glider of mass  $m$ , floating on the air track, and attached to two springs, one on each of its sides, as shown in the diagram below. This one-mass/two-springs system will have an equilibrium position at which neither of the springs is extended or compressed (or, alternately, each spring is extended the same amount). If placed at this equilibrium position (call it  $x=0$ ) with zero velocity, the glider will remain there at rest forever. If the glider is displaced to the right from the equilibrium position by a distance of positive  $x$ , the right spring will be compressed and the left spring extended, both by the distance  $x$ , and there will be a left-directed force on the glider of  $-kx + (-kx) = -2kx$ , assuming both springs obey Hooke’s Law.



As shown in the figure, the left-hand spring is attached to a fixed support, while the right-hand spring is connected via a string to a *wave driver*. The wave driver pulls up and down on the string with a sinusoidal force of frequency  $f$  imposing a small-amplitude *driving force* on the glider-plus-springs system. This driving force then is described mathematically by  $F(t)=F_0 \sin(2\pi ft)$ , where  $F_0$  is the amplitude of the driving force.

Now that we have defined the forces that act on the glider, let's apply Newton's Second Law to this object. Since the spring forces are not constant, but instead depend on the distance  $x$ , we'll have to write Newton's Law in its calculus form

$$F_{\text{net}} = m \frac{d^2x}{dt^2}. \quad [1]$$

Filling in the forces, we find

$$-2kx + F_0 \sin(2\pi ft) = m \frac{d^2x}{dt^2}. \quad [2]$$

The above expression, called a *differential equation*, relates the quantity  $x$  to its second derivative with respect to time. Given advanced calculus techniques, it's possible to show that one solution to this equation is

$$x(t) = A \sin(2\pi ft). \quad [3]$$

This solution tells us that the time-dependent position  $x$  of the glider is described by a sine wave whose frequency is the same as that of the driving frequency  $f$ . Thus the glider's motion is oscillatory, continuously moving back and forth as it repeats the basic cycle of a sinusoid.

Interestingly, the mathematics also predicts that the amplitude  $A$  of this oscillatory motion, that is, the maximum distance that the glider attains from the equilibrium position during its motion, is given by

$$A = \frac{F_0}{4\pi^2 m} \frac{1}{|f^2 - f_0^2|} \quad [4]$$

where

$$f_o = \frac{1}{2\pi} \sqrt{\frac{2k}{m}}. \quad [5]$$

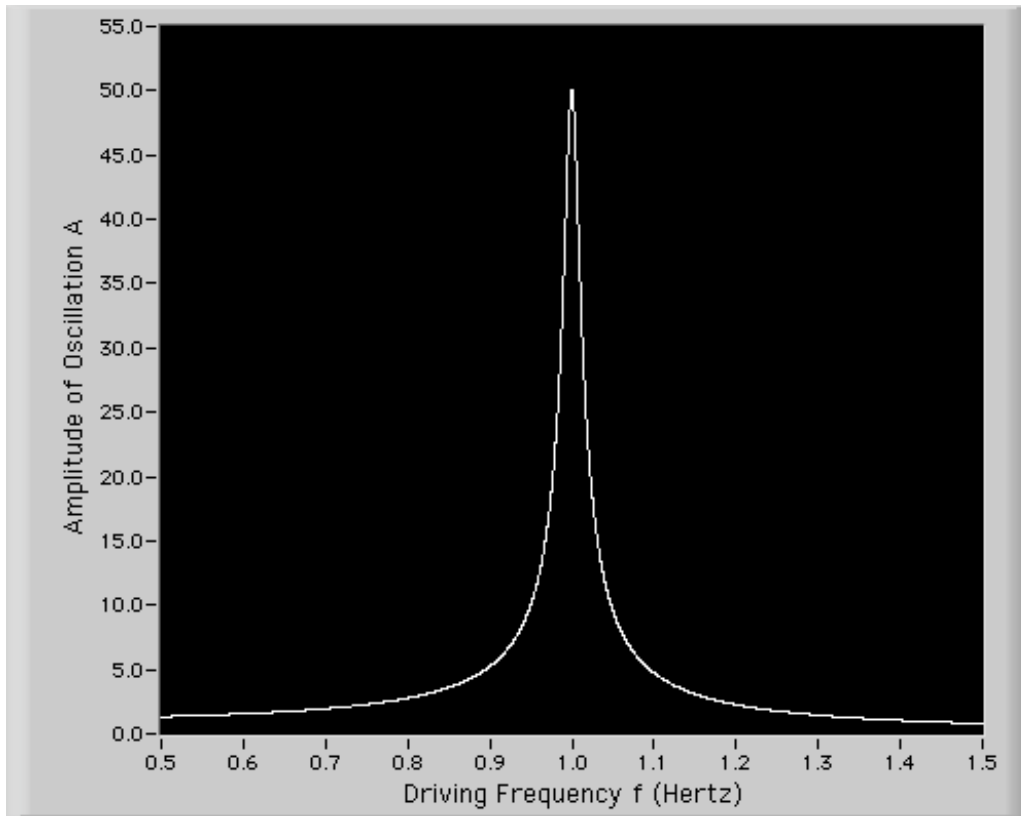
The frequency  $f_o$  is called the “natural frequency” of our one-mass/two-springs system and it is the frequency at which this system will oscillate, in the absence of a driving force.

Equation [4] predicts that the system’s resulting oscillatory amplitude  $A$  approaches zero when the driving frequency  $f$  deviates very far from  $f_o$ . However, Equation [4] also tells us that a driving force is very effective at producing a large amplitude oscillations when its frequency  $f$  is near the natural frequency  $f_o$  of the system. This phenomenon – the large-amplitude oscillations that result when a driving force frequency matches the natural frequency of a system – is called *resonance*.

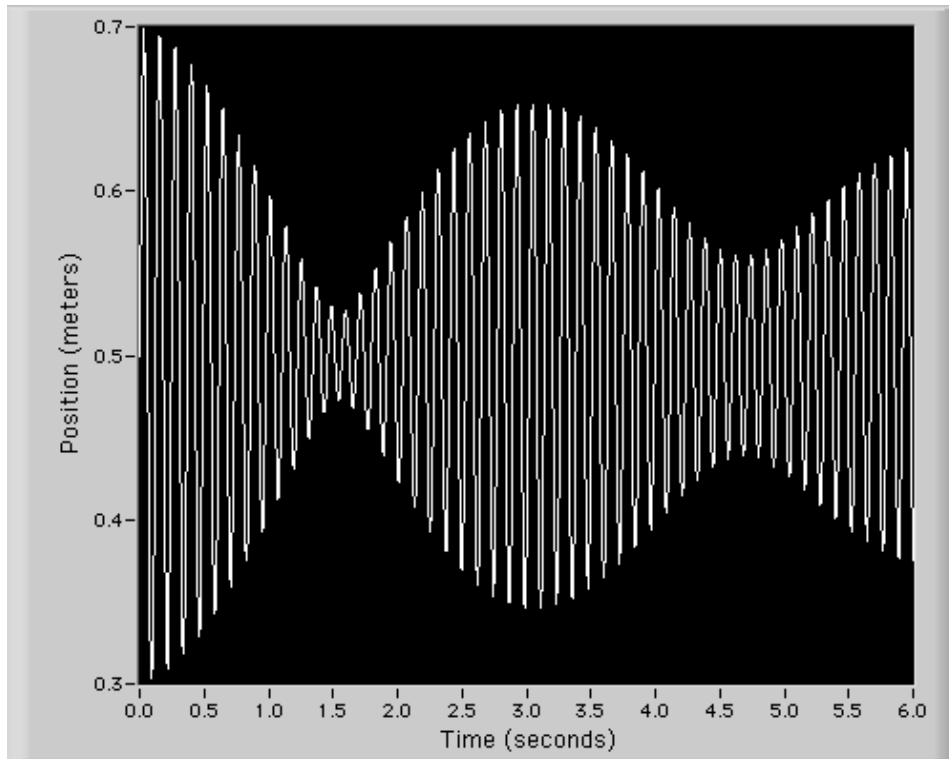
In fact, Equation [4] predicts that the amplitude of oscillation  $A$  will be infinite, if  $f = f_o$ . This prediction is tempered somewhat when one includes frictional effects in the above analysis. This more comprehensive analysis yields the same qualitative picture as above, but gives the following expression for the oscillatory amplitude  $A$  as a function of the driving frequency  $f$ :

$$A = \frac{F_o}{4\pi^2 m} \frac{1}{\sqrt{(f^2 - f_o^2)^2 + \beta^2 f^2}}. \quad [6]$$

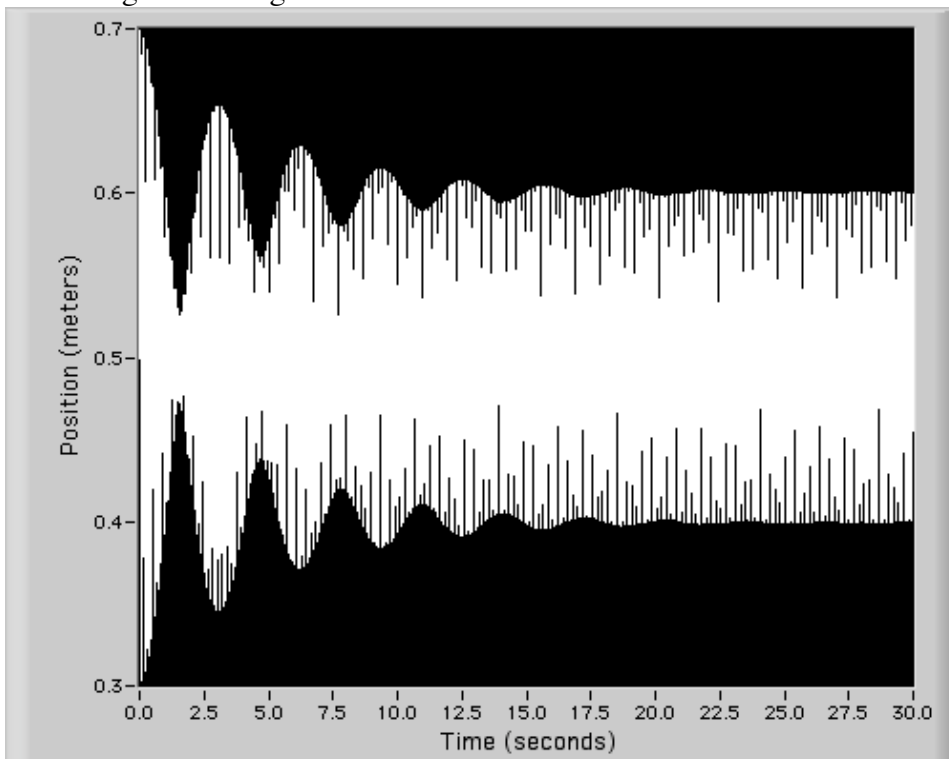
Here the constant  $\beta$  is a measure of the amount of friction acting on the system. In our experiment, a magnet attached to the oscillating mass will allow us to control the size of  $\beta$ . Equation [6] predicts that the amplitude  $A$  drops to zero as  $f$  deviates far from  $f_o$ , but assigns  $A$  the finite value of  $F_o / (4\pi^2 m \beta f_o)$  when  $f = f_o$ . To get a feel for the message contained in this expression for  $A$ , a plot of Equation [6] follows, where we have taken the natural frequency  $f_o = 1$  Hz,  $\beta = 0.02$  and  $F_o / (4\pi^2 m) = 1$ . We see that for this choice of parameters, the system will oscillate with maximum amplitude when the driving frequency equals the system's natural frequency of 1 Hz as expected, but that the system still will oscillate with appreciable amplitudes when the driving frequency is in the range of 0.9 to 1.1 Hz. This plot is called a *resonance curve*.



Finally, one complication in the study of oscillating masses must be mentioned. As derived above, when a sinusoidal driving force of frequency  $f$  is applied to a one-mass/two-springs system, it oscillates at the frequency  $f$ , with an amplitude  $A$  given by Equation [6]. However, the act of turning on the driving force delivers an impulse to system, which causes an additional transitory oscillation at its natural frequency  $f_0$ . The system then oscillates simultaneously at these two frequencies,  $f$  and  $f_0$ , resulting in a net position versus time graph that has a time-varying amplitude as shown on the following plot. This plot shows the response of a certain one-mass/two-springs system during the first 6 seconds after a driving force (whose frequency  $f$  is not equal to  $f_0$ ) was turned on. This resultant pattern, due to the sum of two unequal-frequency sine waves, is a well-known phenomenon in physics and is given the name *beats*.



Thankfully, though, any frictional forces in the system, no matter how small, will act to damp out the bothersome oscillation at the natural frequency as time goes on. So, by waiting an appropriate amount of time, the beats will disappear. The following plot shows the response of the same system as above, but now over the 30-second time interval following the driving force turn-on.



The individual cycles of oscillation cannot be resolved on this longer time-scale plot, but it is clear that after about 20 seconds waiting time, the beat pattern (i.e., the time-varying amplitude of oscillation) has disappeared, indicating that the transient oscillation at the system's natural frequency has been completely extinguished by friction. Thus, for this particular system, one can have confidence that, after a 20-second waiting time, the amplitude of oscillation is then due completely to the driving force and so make a measurement of the amplitude  $A$  predicted by Equation [6].

## Experiment

### I. Determining the Spring Constant

The force exerted by a stretched string is given by  $F = -kx$  where  $x$  is the distance stretched and  $k$  is the *spring constant*. Determine  $k$  by hanging several different masses from the spring and measuring the distance that each mass causes the spring to stretch. Then plot  $F (=mg)$  vs.  $x$ ; the slope of the best-fit line is the spring constant. You should take at least 5-6 data points, using masses between 20 g and 200 g.

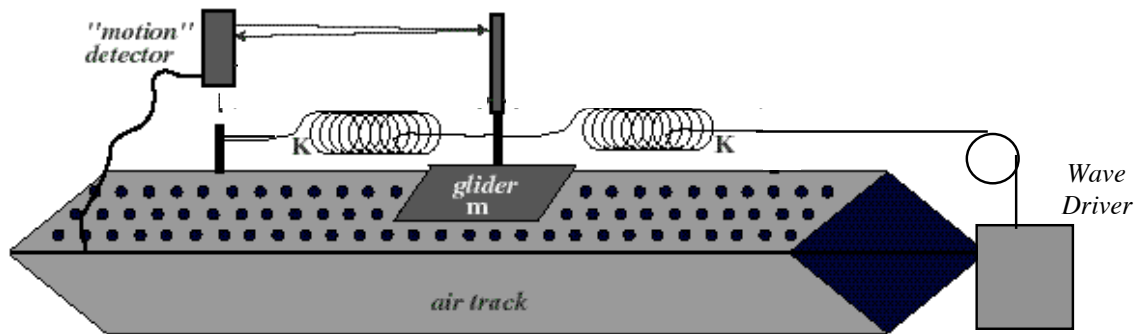
### II. Single Oscillating Mass

In this investigation, you will use the Physics 100 lab's computer-based data acquisition tools to study the oscillations of a single glider of mass  $m$  connected to two springs. A magnet will be attached to one side of the glider so that a retarding force of controllable magnitude acts on the glider as it moves along the air track. This retarding force results from Faraday's Law (one of the basic laws of electromagnetism that we will study next semester) and it has the characteristics of a velocity-dependent frictional force.

Attach the magnet holder to the metal pin on one side of the glider. The constant  $\beta$  for the oscillating system (see Equation [6]) is determined by the position of the magnet relative to the aluminum air track ( $\beta$  is largest when the magnet is closest to the track). For best results, position the magnet holder either near the middle or top of the metal pin. Balance the glider by adding an equal mass to the pin on the other side of the glider.

Determine the total mass  $m$  of the glider including its attached "sail" (see below), magnet holder and balancing mass using the lab's Digital Balance. Then, with your knowledge of the springs' spring constant  $k$  from Part I of this experiment, use Equation [5] to predict the natural frequency  $f_0$  of this oscillatory system.

Turn on the air blower for the air track. Place a single glider on the track, and attach springs to both of its sides. As shown in the diagram below, the "non-glider" end of one spring should be attached to a fixed post, while the "non-glider" end of the other spring is connected (via a string and pulley) to the wave driver. Taking care not to over-extend (and thus damage) the springs, slide the fixed post away from the pulley until each spring is elongated and remains taut even as the glider is displaced about 10 cm in either direction from its equilibrium position. Secure the fixed post to the air track.



**WARNING: Always turn on the Power Amplifier before launching DataStudio. Turning on the Power Amplifier while DataStudio is open may cause a power surge, resulting in damage to your computer.**

Configure the computer-based data acquisition tools for your investigation. As shown in the figure above, the glider has a “sail” attached on top to act as a reflecting surface for ultrasound so that the position of the glider can be monitored by a Motion Sensor. In Data Studio’s **Experiment Setup** window, add a **Motion Sensor** to the **Digital Channels** input, click on the Motion Sensor icon to open the Motion Sensor **Properties** window, and set the **Sample Rate** to 50. This selection instructs the Motion Sensor to acquire 50 position measurements per second, but be aware that it limits the detector to a maximum range of about 1 meter. Click the **Motion Sensor** button and the Motion Sensor will acquire distance measurements and display them in the **Present Sensor Distance** indicator. Position the Motion Sensor on a stand and orient it at a proper angle so that its ultrasonic signal properly reflects from the glider's sail and the unit gives an accurate measurement of the glider's position. If your experimental configuration ends up requiring a longer Motion Sensor range than 1 meter, simply decrease the **Sample Rate** accordingly. Once you are satisfied with your configuration, create a **Graph** to monitor the **Position** measurements of the Motion Sensor.

The wave driver is controlled by an attachment called the *Power Amplifier*, whose circular-shaped DIN plug must be connected to one of the **Analog Channels** inputs of the *ScienceWorkshop Interface*.

In DataStudio’s **Experiment Setup** window, click the appropriate Analog Channel and select **Power Amplifier** from the list. The **Signal Generator** window will then appear on your computer screen (if it remains hidden, click on the **Output** icon on the *ScienceWorkshop Interface*). Once attached to the wave driver via two banana-plug cords, the Power Amplifier can output an electronic sine wave of selectable frequency and amplitude, which then is converted to a mechanical sine-wave motion by the wave driver. Within the **Signal Generator** window, try inputting 1.00 for **Frequency** and 5.0 volts for **Amplitude**, click the **Auto**, then **ON** button. The wave driver should then start moving its top post up and down at a frequency of 1 complete cycle per second (= 1 Hz).

Here are three possible things to investigate:

### A. Natural Frequency:

Above, you used Equation [5] to predict the natural frequency  $f_0$  of the one-mass/two-springs system. Using the **Signal Generator**, program the wave driver to drive the system at this predicted frequency, with **Amplitude** set to 5 Volts (if you choose too large of an amplitude, the sine-wave output by the Power Amplifier will be distorted and a red-light indicator will illuminate to prompt you to reduce the amplitude). You should observe the glider start to oscillate with a large amplitude and be able to monitor this oscillation on the **Graph** displaying the Motion Sensor's *Position versus Time* measurements.

Try making slight changes to the driving frequency  $f$  to find the value that produces the maximum amplitude of oscillation. When you find the magic value for  $f$  that gives maximal oscillatory amplitude, then that  $f$  equals  $f_0$ . One trick you can use to help you with this tweaking: If, when you turn on a given driving force, you observe a beat pattern on the graph of *Position vs. Time*, that driving force does not equal  $f_0$ . When  $f$  is very close to  $f_0$ , the beats (i.e., the time-variation of the oscillatory amplitude) will become very slow. The beat pattern will disappear when  $f = f_0$ .

How well does the value for  $f_0$  predicted by Equation [5] match the value for  $f_0$  that you determined empirically?

### B. Resonance Curve:

Determine the resonance curve for your system by measuring the oscillatory amplitude  $A$  for several driving frequencies  $f$  that bracket  $f_0$ . For example, if  $f_0 = 0.9$  Hz, you might want to measure  $A$  for several values of  $f$  in the range of 0.7 to 1.1 Hz.

Each time you apply a new driving frequency  $f$  to the system, you must wait sufficiently long before measuring the resultant amplitude  $A$  so that frictional effects can damp out the unwanted transitory oscillation at the natural frequency  $f_0$ . Thus, it is desirable to have a certain amount of friction acting on the glider, and so we have attached a magnet to the glider (which produces a friction-like force via Faraday's Law) for this purpose. By monitoring the beat pattern on the Motion Sensor's *Position versus Time* graph, you can determine the proper waiting period for your system (it will probably take a minute or so).

For a given value of  $f$ , once the beat pattern has disappeared, determine the distance between the peak and trough of the sine-wave oscillation displayed on the *Position versus Time* graph. By definition, the amplitude  $A$  is half of this peak-to-trough distance.

With the above procedure, generate a table of driving frequencies  $f$  and their associated oscillation amplitudes  $A$ . In **DataStudio**, select **New Empty Data Table...** from the **Experiment** menu, enter your  $A$  (y-quantity) vs.  $f$  (x-quantity) data and then plot the resonance curve  $A$  (y-axis) vs.  $f$  (x-axis). Save this work in a file on your home

server. Would you characterize your curve as very sharp (i.e., has significant amplitudes only over a very small range of f-values) or as broad? It can be shown from Equation [6] that the width of this curve is determined by the amount of friction in the oscillating system (larger friction produces a wider curve).

### C. Mathematical Description of Resonance Curve:

The shape of your resonance curve should be described by Equation [6]. To demonstrate this fact, first note that Equation [6] can be written as

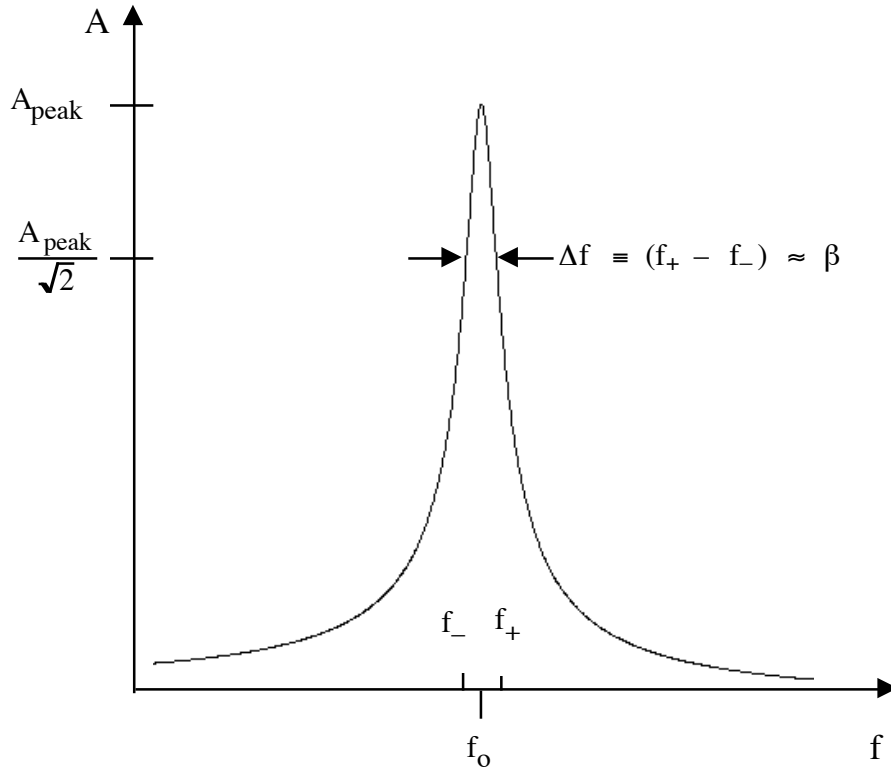
$$A = \frac{\alpha}{\sqrt{(f^2 - \gamma^2)^2 + \beta^2 f^2}} \quad [8]$$

where the constants  $\alpha \equiv F_0/4\pi^2 m$  and  $\gamma \equiv f_0$ . With access to sophisticated data analysis software, which can perform a least-squares fit of data to a user-supplied function, you can program this software to fit your resonance curve data to the functional form of Equation [8]. The fitting routine will then output the "best-fit" values for  $\alpha$ ,  $\beta$  and  $\gamma$  and may even automatically plot the "best-fit" curve for you to compare with your data points. Unfortunately, neither **DataStudio** nor **Excel** is capable of performing this task.

Another approach for determining  $\alpha$ ,  $\beta$  and  $\gamma$  from your data is as follows:

- Gamma: by definition,  $\gamma \equiv f_0$ , so simply set  $\gamma$  equal to the value of the system's natural frequency that you determined empirically.
- Beta: The resonance curve's peak amplitude  $A = A_{\text{peak}}$  occurs when  $f = f_0$ . As seen in the next diagram, there are two frequencies, which we define to be called  $f_+$  and  $f_-$ , which produce an oscillatory amplitude  $A$  that is  $1/\sqrt{2} = 0.707$  times the peak value. That is, when  $f = f_+$  or  $f = f_-$ ,  $A = A_{\text{peak}}/\sqrt{2} = 0.707 A_{\text{peak}}$ . Plugging these defined frequencies and their associated amplitudes into Equation [8], and assuming that frictional effects are small so that  $\beta \ll f_0$ , it is possible to show that the difference between these frequencies is approximately equal to  $\beta$ . That is,

$$\Delta f \equiv (f_+ - f_-) \approx \beta \quad [9]$$



Estimate  $f_+$  and  $f_-$  from your resonance curve data, then use Equation [9] to determine  $\beta$ .

- Alpha: The resonance curve's peak amplitude  $A = A_{\text{peak}}$  occurs when  $f = f_0 = \gamma$ . Putting these values into Equation [8] we find

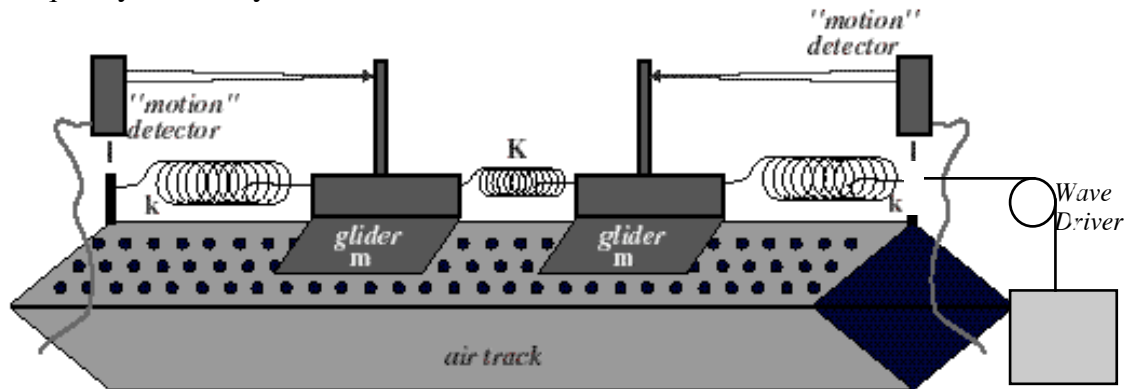
$$A_{\text{peak}} = \frac{\alpha}{\beta\gamma} \Rightarrow \underline{\underline{\alpha = \beta\gamma A_{\text{peak}}}}. \quad [10]$$

Determine  $A_{\text{peak}}$  from your resonance curve data, then with your values for  $\beta$  and  $\gamma$ , find  $\alpha$  using Equation [10].

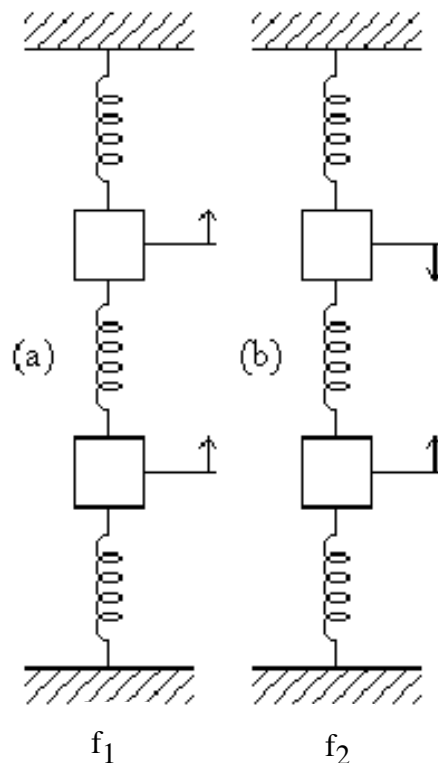
In **DataStudio**, plot Equation [8] with your determined values for  $\alpha$ ,  $\beta$  and  $\gamma$  and compare this curve with your plotted  $A$  vs.  $f$  data points. To make this comparison, first graph your  $A$  vs.  $f$  data, then in the Graph's toolbar, click on **Fit** and select **User Defined**. In the dialog window that appears, program Equation [8] in the box to the right of **User Defined** and define the values for  $\alpha$ ,  $\beta$  and  $\gamma$  in the **Coefficients** box. Note that DataStudio always uses  $x$  for the independent variable, and that you should not write a "=" before the equation. (If the **Curve Fit** window does not appear, double click on **Fit Statistics** on the graph window.) Does your resonance curve data obey Equation [8] fairly well? If your estimated values for  $\alpha$ ,  $\beta$  and  $\gamma$  do not appear optimal (that is, they are not the "best-fit" values), try tweaking with the values in the **Coefficients** box until you find the best agreement between the plot of Equation [8] and the experimental data.

### III. Two Oscillating Masses Coupled by a Spring

Using three identical springs and two gliders equipped with "sails," construct the following set-up to investigate the oscillatory patterns of two masses, each of mass  $m$ , coupled with springs. Configure the computer-based tools to display simultaneous plots of *Position versus Time* for both gliders as the Wave Driver applies a driving force of frequency  $f$  to the system.



By applying Newton's Second Law to each of the masses in the two-mass/three-springs system, two differential equations result. These equations can be solved using the techniques of calculus, and the solution predicts two natural patterns of oscillations for this system, each with its own distinct natural frequency. The character of these two vibrational modes is shown in the diagram below.



A short description of each of these vibrational modes follows:

- **Symmetric Mode:** The mode labeled (a) in the diagram is the symmetric vibrational mode. In this mode, both masses, each of mass  $m$ , move in the same direction at all times (i.e., they are "in phase") with the same amplitude and the frequency  $f_1$  of their oscillation is given by

$$f_1 = \frac{1}{2\pi} \sqrt{\frac{k}{m}}. \quad [11]$$

- **Antisymmetric Mode:** The mode labeled (b) in the diagram is the antisymmetric vibrational mode. In this mode, both masses move in opposite directions at all times (i.e., they are "180° out of phase") with the same amplitude and the frequency  $f_2$  of their oscillation is given by

$$f_2 = \frac{1}{2\pi} \sqrt{\frac{3k}{m}}. \quad [12]$$

By comparing Equations [11] and [12], we see that the antisymmetric mode occurs at a frequency that is  $\sqrt{3}$  times higher than the frequency of the symmetric mode. A sharp resonance curve is centered on each of these frequencies.

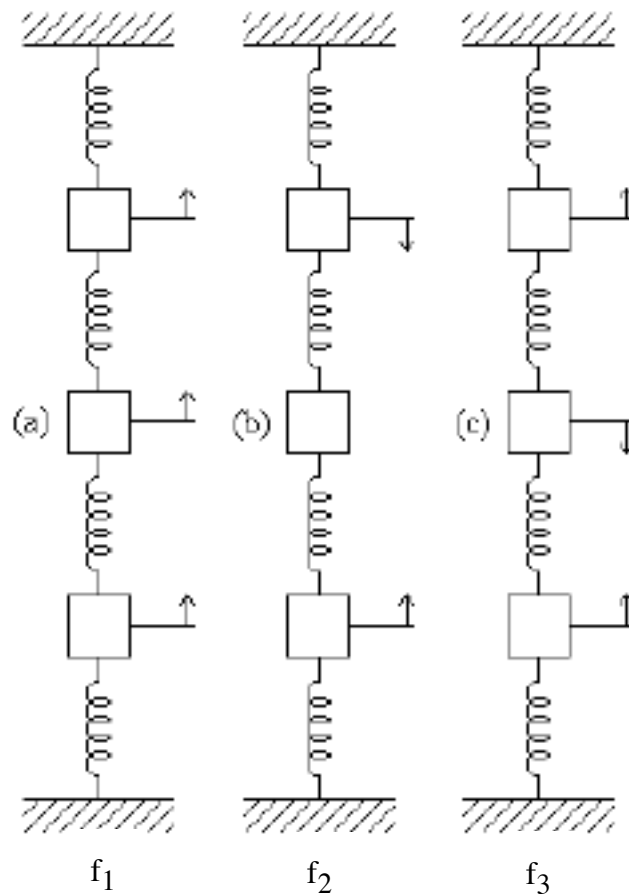
Experimentally investigate the two-mass/three spring system. With your knowledge of  $k$  and  $m$ , use Equations [11] and [12] to predict the natural frequencies  $f_1$  and  $f_2$ . Excite these modes experimentally and compare your empirically determined values for  $f_1$  and  $f_2$  with the values you predicted theoretically. For both the symmetric and antisymmetric mode, compare the *Position versus Time* graph acquired for each mass. Do the masses oscillate in phase (180° out of phase) with equal amplitudes in the symmetric (antisymmetric) mode?

#### IV. Three Oscillating Masses Coupled by Springs

Using four identical springs and three gliders, construct a set-up to investigate the oscillatory patterns of three masses, each of mass  $m$ , coupled with springs. Configure the computer-based tools to display simultaneous plots of *Position versus Time* for two gliders at a time as the Wave Driver applies a driving force of frequency  $f$  to the system.

By applying Newton's Second Law to each of the masses in the three-mass/four-springs system, three differential equations result. When these equations are solved using the techniques of calculus, three natural patterns of oscillations are predicted for this

system, each with its own distinct natural frequency. The character of these three patterns is shown in the diagram below.



A short description of each of these vibrational modes follows:

- Low-Frequency Mode:** The mode labeled (a) in the diagram is lowest frequency vibrational mode. In this mode, all three masses, each of mass  $m$ , move in the same direction at all times (i.e., they are "in phase") The two outside masses move with the same amplitude, while the amplitude of the central mass is greater by a factor of  $\sqrt{2}$ . The oscillatory frequency  $f_1$  of this mode is given by

$$f_1 = \frac{1}{2\pi} \sqrt{\frac{(2 - \sqrt{2})k}{m}}. \quad [13]$$

- Intermediate-Frequency Mode:** In the mode labeled (b) in the diagram, the central mass remains stationary, while the two outside masses oscillate with the same

amplitude, but in opposite directions at all times (i.e., they are “180° out of phase”).

The frequency  $f_2$  of their oscillation is given by

$$f_2 = \frac{1}{2\pi} \sqrt{\frac{2k}{m}}. \quad [14]$$

- **High-Frequency Mode:** In the mode labeled (c) in the diagram, the two outside masses oscillate in the same direction (i.e., in phase with each other), while the central mass oscillates in the opposite direction (i.e., 180° out of phase with outside masses). The two outside masses move with the same amplitude, while the amplitude of the central mass is greater by a factor of  $\sqrt{2}$ . The frequency  $f_3$  of the masses is given by

$$f_3 = \frac{1}{2\pi} \sqrt{\frac{(2 + \sqrt{2})k}{m}}. \quad [15]$$

Experimentally investigate the three-mass/four spring system. With your knowledge of  $k$  and  $m$ , use Equations [13], [14] and [15] to predict the natural frequencies  $f_1$ ,  $f_2$  and  $f_3$ . Excite these modes experimentally and compare your empirically determined values for  $f_1$ ,  $f_2$  and  $f_3$  with the values you predicted theoretically. For all three modes, acquire *Position versus Time* data for each mass (you'll have to improvise a little here, because you will only be able to acquire this data “two masses at a time”). Do the masses oscillate with the expected relative amplitudes and phase relations (that is, in phase or 180° out of phase)?