Note on sinc-kernel sums and Poisson transformation

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In a recent paper [2] R. Baillie, D. Borwein and J. Borwein pose various interesting problems such as that of evaluating sums they define

$$s_{N,M} := \sum_{n=1}^{\infty} (\mathrm{sinc}\ n)^N \cos^M n,$$

where, standardly, sinc $z := (\sin z)/z$. Herein we show how to evaluate these, and a great many other "sinc-kernel sums," via Poisson transformation. Coincidentally, Poisson summation just recently was a key expedient in a separate experimental-mathematical treatment to resolve some absolute constants [1].

Bypassing issues of convergence, a general Poisson formula for summation over integer indices is

$$\sum_{n \in Z} f(n) = \sum_{\mu \in Z} \int_{-\infty}^{\infty} f(x)\ e^{2\pi i \mu x}\ dx.$$

Thus, a certain summation with a sinc-power kernel is, at least formally,

$$F(N,k) := \sum_{n \in Z} (\mathrm{sinc}\ n)^N \cos(kn) = \sum_{\mu \in Z} \int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^N \ e^{i(k+2\pi\mu)x}\ dx.$$

In this sense, to get many different sinc-kernel sums *it suffices to know the Fourier transform of a power of sinc*, that is to know

$$\mathcal{F}(N,\omega) := \int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^N \ e^{i\omega x}\ dx.$$

But in turn, this transform is known in finite form, for any positive integer N, as[1]

$$\mathcal{F}(N,\omega) = \frac{\pi}{2^N (N-1)!} \sum_{j=-\lceil N/2 \rceil}^{\lfloor N/2 \rfloor} \binom{N}{j + \lceil N/2 \rceil} (\omega - 2j - \delta_{N\mathrm{odd}})^{N-1} \mathrm{sgn}(\omega - 2j - \delta_{N\mathrm{odd}}) (-1)^{j + \lceil N/2 \rceil}.$$

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[1]We omit the rather tedious details; but see the ensuing points (1)-(3) which in themselves contain the seeds of derivation.

Some digressional remarks are in order here:

1) This finite-form Fourier transform $\mathcal{F}$ is foreshadowed already in the referenced work [2], where Bernoulli polynomials appear. In a sense the present Fourier approach "unrolls" the Bernoulli identities combinatorially.

2) It is interesting that because the sinc function is—up to a constant factor—the Fourier transform of a "pedestal", or pulse-function on $(-1, 1)$, the $\mathcal{F}$ with argument N is (2π) times the probability density of the random walk after N incremental-pedestal jumps. This thinking should lead to a version of the celebrated central-limit theorem in regard to sinc-kernel sums. That is to say, we should have, in some asymptotic sense,

$$\mathcal{F}(N, \omega) \sim \sqrt{\frac{3\pi}{N}} e^{-3\omega^2/(4N)},$$

with an ensuing asymptotic estimate on, say, the sums $s_{N,M}$.

3) Beyond the $s_{N,M}$ of the previous authors, it is likewise possible to consider

$$\sum_{n \in Z} \operatorname{sinc}^N(n) g(n)$$

for more general g, and combine the Fourier transforms properly, to obtain closed forms.

On point (3) above, one could in principle invoke a Fourier transform of $\cos^M x$ as a superposition of delta-functions. In this special case, though, it is simpler just to observe the finite, elementary decomposition

$$\cos^M x = 2^{-M} e^{iMx} \sum_{m=0}^{M} \binom{M}{m} e^{-2imx},$$

then insert such into the defining sum for $s_{N,M}$, to arrive quickly at a representation for the Baillie–Borwein–Borwein sinc-sum, namely

$$s_{N,M} = -\frac{1}{2} + \frac{1}{2^{M+1}} \sum_{m=0}^{M} \binom{M}{m} \sum_{\mu \in Z} \mathcal{F}(N, M - 2m + 2\pi\mu).$$

However, the formally infinite sum over μ is *actually a finite sum* because of the sgn functions in the Fourier transform above. We have thus achieved a finite form for the Baillie–Borwein–Borwein sum.

In this way we have also proved the theorem, typical of the original treatment [2], that every $s_{N,M}$ is a rational polynomial in π.

Next appears an example matrix of values $1 + 2s_{N,M}$. As the matrix shows, those original authors correctly detected experimentally that various sets of s values coincide; and yet, as they also surmise for related sums, long polynomials in π start emerging.

$$\begin{pmatrix} \frac{\pi}{2} & \frac{\pi}{2} & \frac{3\pi}{8} & \frac{3\pi}{8} \\ \frac{\pi}{2} & \frac{\pi}{2} & \frac{3\pi}{8} & \frac{3\pi}{8} \\ \frac{\pi}{2} & \frac{7\pi}{16} & \frac{3\pi}{8} & \frac{71\pi}{64} - \frac{7\pi^2}{16} + \frac{\pi^3}{16} \\ \frac{23\pi}{48} & \frac{5\pi}{12} & \frac{413\pi}{192} - \frac{49\pi^2}{32} + \frac{7\pi^3}{16} - \frac{\pi^4}{24} & \frac{5\pi}{3} - \pi^2 + \frac{\pi^3}{4} - \frac{\pi^4}{48} \end{pmatrix}$$

Here, the rows are labeled $N = 1, 2, 3...$ downward and the columns are labeled $M = 1, 2, 3$ rightward.

References

[1] D. Bailey, J. Borwein, and R. Crandall, "Resolution of the Quinn-Rand-Strogatz constant of nonlinear physics," preprint June 2007.

[2] Robert Baillie, D. Borwein, and Jonathan M. Borwein, "Surprising Sinc Sums and Integrals," *Amer. Math. Month.*, to appear, 2007.