On a Bessel-integral of J. Borwein

R. E. Crandall
24 Aug 2004

J. Borwein has asked for a rapid numerical evaluation scheme for the integral

$$\mathcal{J} := \int_0^\infty \int_0^\infty \frac{J_1^2(2\sqrt{xy})}{(e^x - 1)(e^y - 1)}\, dx\, dy.$$

It will not do to attempt a straightforward insertion of the expansion

$$J_1^2(t) = \sum_{n=1}^\infty a_n t^{2n},$$

even though one can derive the exact coefficient

$$a_n = \frac{(-1)^{n-1}}{4^n n!^2} \binom{2n}{n+1}.$$

Because

$$|a_n| \sim \frac{1}{n!^2 \sqrt{\pi n}}$$

it turns out that the formal integral for $\mathcal{J}$—with the power series simply inserted—does separate into disjoint x, y integrals, but gives a divergent expansion in terms of Riemann-ζ functions.

Happily, the asymptotic character of the a_n allows us to develop a finite part of the $\mathcal{J}$-integral, and then use formal power-series insertion. To this end, define

$$H(m, n) := \int_0^\infty \int_0^\infty J_1^2(2\sqrt{xy})\, e^{-mx-ny}\, dx\, dy.$$

Now, formally, we have the Borwein–Bessel integral as

$$\mathcal{J} = \sum_{m,n \geq 1} H(m, n),$$

and we also have the exact evaluation (via symbolic manipulation, say):

$$H(m,n) = \frac{1}{2}\left(\frac{mn+2}{\sqrt{mn(mn+4)}} - 1\right).$$

It turns out that if we truncate the formal sum over the $H(m,n)$ and then insert the a_n series, convergence of the integrated series is assured. Indeed, choosing a cutoff integer N, we have formally

$$\mathcal{J} = \left(\sum_{m=1}^{N-1}\sum_{n=1}^{N-1} + \sum_{m=N+1}^{\infty}\sum_{n=1}^{\infty} + \sum_{m=1}^{N}\sum_{n=N+1}^{\infty}\right) H(m,n).$$

Now it turns out that for sufficiently large N, say $N > 4$, the insertion of the a_n series into the H integral gives absolutely convergent series for the 2nd and 3rd sums.

The final N-cutoff expansion is, assuming the closed-form $H(m,n)$ and the Hurwitz zeta function $\zeta(s,N)$ such that $\zeta(s,1) = \zeta(s)$,

$$\mathcal{J} = \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} H(m,n) + \sum_{s=2}^{\infty}(-1)^s \binom{2s-2}{s}\left(2\zeta(s)\zeta(s,N) - \zeta(s,N)^2\right).$$

Taking the threshold case $N = 5$, we have

$$\mathcal{J} = -8 + \frac{9}{4\sqrt{2}} + \frac{5\sqrt{3}}{4} + \frac{21}{8\sqrt{5}} + \frac{5}{2\sqrt{6}} + \frac{11}{6\sqrt{13}} + \frac{4}{\sqrt{15}} + \frac{5}{\sqrt{21}} +$$
$$\sum_{s=2}^{\infty}(-1)^s \binom{2s-2}{s}\left(2\zeta(s)\zeta(s,5) - \zeta(s,5)^2\right),$$

which is an algebraic sum plus a geometrically convergent "tail," since $\zeta(s,N) \sim 1/N^s$. Taking $N = 40$ allows about one decimal digit per summand of the zeta-tail, to yield such as

$$\mathcal{J} \approx 1.1038396536176132500751953873457081344493477394099807622387.$$